The King's Court

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Intro

In June 2023, I went to the SIAM Conference on Optimization to speak about recent progress towards Woodall's conjecture. I attended many talks at the conference, but by far my favourite was by Professor Lisa Hellerstein on the Stochastic Boolean Function Evaluation problem.

In it, she provided a puzzle with a solution so elegant that I immediately gave the problem as a riddle to all my friends. The following situation is a bit more dramatic than the dry world of boolean functions, but it captures the same mathematical essence.

Riddle

You are the king of a small province where you preside over a court of n nobles. Harmony reigns in your land for 10 peaceful years, when one day you hear rumours that the court is plotting a violent coup to overthrow your rule.

You are flabbergasted! Why would the court want to overthrow a benign ruler such as yourself? You decide that you must personally verify whether the rumours are true by interrogating a subset of the court members.

Some court members are more suspicious than others. For each court member $i \in \{1, \ldots, n\}$, the probability that they are planning to overthrow you is p_i.

You know that if at least k court members are plotting to overthrow you, then the coup will happen tomorrow. On the other hand, if fewer than k court members are plotting to overthrow you, then you are safe.

You scratch your chin and pace about, deep in thought. On the one hand, it is imperative to determine with certainty whether a coup will take place. On the other hand, it would be preferable for the good of the kingdom to keep interrogations to a minimum.

You decide to consult your academic advisor, who is wise in the ways of mathematics:

Question: What is the optimal (adaptive) sequence of interrogations so that the king can verify whether a coup will take place using the minimum expected number of interrogations?

Answer

Hint 1: Consider the case that you know a coup will take place. That is, you know that there are at least k treacherous court members, but you are still tasked with verifying this fact in the minimum (expected) number of interrogations. What is the optimal sequence of court members to examine?

Hint 2: Consider the case that you know a coup will *not* take place. That is, you know that there are at least $n-k+1$ loyal court members, but you are still tasked with verifying this fact in the minimum (expected) number of interrogations. What is the optimal sequence of court members to examine?

Hint 3: Is there a choice of first court member to examine which is optimal in both cases?

Answer: Sort the court members $\{x_1, \ldots, x_n\}$ in non-increasing order of p_i, so that the first court member x_1 is the most likely to be treacherous, and the last x_n is most likely to be loyal. In the case of the first hint, it is optimal to interrogate the court members in this order. This is natural since, in this case, we are trying to find k treacherous court members as quickly as possible.

On the other hand, in the case of the second hint, we are seeking to find $n-k+1$ loyal court members as quickly as possible. Hence, an optimal scheme is to interrogate the court members in the reverse order $x_n, \ldots, x_1$ until enough loyal members are found.

However, let's analyze this a bit more carefully. Consider the first case, in which there are at least k treacherous members. No matter what the outcome of my interrogations, I must interrogate at least k people before verifying this fact. Hence, it is still optimal in this case to interrogate x_k first, and then proceed in the order $x_1, \ldots x_{k-1}, x_{k+1}, \ldots, x_n$. Similarly, in the second case, I will always interrogate at least $n-k+1$ members. Lo and behold, it is still optimal to select x_k to be interrogated first!

Thus, in either case it is optimal to interrogate x_k first. Our next action then depends on the outcome of this interrogation. If court member x_k is treacherous, we are faced with a new instance of the same problem with $n-1$ remaining court members, $k-1$ of whom must be treacherous for a coup to occur. If, on the other hand, x_k is loyal, then again we are faced with a new instance with $n-1$ members, k of whom are needed to execute a coup.

In either case, we can perform the same reasoning as above to obtain the next optimal member to interrogate, and repeat this process until we are convinced of our safety . . . or our demise.

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